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A LOWER MULTINOMIAL BOUND FOR THE TOTAL OVERSTATEMENT ERROR IN ACCOUNTING POPULATIONS*

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A lower bound on the total error in an accounting population is required, in conjunction with the point estimate of the total error amount and the upper bound, when adjusting an account to determine the amount of the adjustment. This paper extends the multinomial methodology for obtaining an upper bound on the total overstatement (or understatement) error in an accounting population to the determination of a lower bound on the total overstatement (or understatement) error. The methodology for obtaining a lower multinomial bound differs in several important respects from that for obtaining an upper bound. The proposed lower bound may be computed for up to 25 errors in the sample and provides tighter limits than the widely used Stringer bound.

(ACCOUNTING/ADJUSTMENTS—MODELING; NONLINEAR OPTIMIZATION—ALGORITHM DEVELOPMENT)

Introduction

In many accounting situations, managers, auditors, and third parties need to adjust account balances to more accurately reflect the true state of the account. The account of interest may comprise an entire file for inventory or accounts receivable, or an individual record for a single customer. Audits leading to adjustments are conducted by independent accountants involving the financial statements of a firm, by governmental accountants involving federal grants and contracts with university researchers, private or corporate income tax returns, Medicare and Medicaid programs, and defense contracts, and by internal auditors involving the firm's accounting process.

Various methods of variables sampling procedures have been proposed for sampling the line items in an account to ascertain information about the total error amount in the account. Estimators such as the mean per unit or ratio and difference estimators lead to a dollar amount point estimate of the total error amount as well as to one or two-sided confidence intervals. If the manager, auditor, or third party finds that the upper limit of the confidence interval for the total error amount exceeds a predetermined level of materiality, he or she can conclude that material errors in the account may be present. When this is the case, an adjustment is often made.[1] The basic issue then is how much should be the amount of the adjustment. One approach is to utilize the point estimate of the total error amount as the amount of the adjustment. This approach uses the "best" estimate of the total error amount, but does not provide recognition that a range of uncertainty exists about the total error amount. Another approach is to make the adjustment equal to the difference between the upper confidence limit for the total error amount and the materiality level. With this approach, the upper confidence limit for the total error amount equals materiality after the adjustment. This approach ignores information from the lower bound of the confidence interval which provides a minimum value for the total error amount. A third approach calls for making an adjustment which is at least as large as the lower

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[1] If a client or a firm is unwilling to adjust, additional sampling is often undertaken to gather more evidence before an adjustment is made.

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confidence interval bound for the total error amount. With this approach, however, the upper limit of the confidence interval for the total error amount after adjustment may still exceed materiality. Clearly, some combination of these approaches is desirable so that the confidence limits after adjustment do not reflect a possible material error and at the same time recognize the minimum total error in the account. Often the adjustment amount is determined by negotiation. In any case, however, determination of an appropriate adjustment amount requires a lower confidence interval bound for the total error amount, as well as the point estimate and upper confidence interval bound.

Unfortunately, the problem of determining the adjustment amount is compounded as a result of repeated findings by Kaplan (1973), Neter and Loebbecke (1975), and others that large-sample confidence bounds based on ratio and difference estimators, which are widely used in practice, are frequently not appropriate for sample sizes commonly used in auditing when the error rate in the population is small. This is frequently the situation in practice. One alternative approach to the use of large-sample confidence bounds developed by Stringer (1963), Anderson and Teitlebaum (1973), and Leslie, Teitlebaum, and Anderson (1979) is to employ dollar unit sampling and upper and lower bounds which are based on: (1) increments in the upper or lower bound for a binomial parameter p for increasing numbers of errors in the sample, and (2) the amounts of the taintings (i.e., prorated dollar errors) found in the sample. The Stringer upper and lower bounds were proposed to avoid reliance on large-sample theory procedures. The Stringer bounds are widely used by most of the large accounting firms but have been shown to be highly conservative.

The conservatism of the Stringer bounds has some potential disadvantages for auditors in that the upper bound will often exceed the prespecified materiality level when the total overstatement (or understatement) errors in the population are less than material, as shown by Duke (1980), leading to costly additional audit work. A tighter upper bound is less likely to indicate the possible presence of a total material error in the population when the total error amount is not material, while at the same time it may still be expected to provide adequate protection in the case when the total error amount in the population is material.

The upper multinomial bound was developed by Fienberg, Neter, and Leitch (1977) to avoid the conservatism of the Stringer upper bound, but a lower multinomial bound has not been available up to this time, even though it is essential for determining an appropriate adjustment amount. In this paper, we will set forth a procedure for obtaining a lower multinomial bound on the total overstatement (or understatement) error in a population.[2] The procedure parallels the one for the upper multinomial bound, yet it differs in some important respects. We first review the upper multinomial bound and then outline the procedure for obtaining a lower multinomial bound. We then explain various procedures to conduct the required optimization efficiently, and compare the effectiveness of the lower multinomial bound with the widely used Stringer bound.[3] Next we consider a clustering procedure for computing a lower bound for larger numbers of errors and examine the effectiveness of this procedure by comparing the results with the Stringer bound for up to 25 errors.

[2]The multinomial bound model may also be applied to other problems where most of the population elements have a value of zero, and the remaining elements have positive values, for instance, the determination of confidence bounds on the level of carcinogens in animals and in air pollutants.

[3]For accounting populations with small error rates, the DUS-Cell bound proposed by Leslie et al. (1979) can also be used to obtain a lower bound on total overstatement error when stratified selection of dollar units is employed. We, however, do not consider the DUS-Cell bound in this paper because of the more restricted sampling scheme for which it was designed.

The Upper Multinomial Bound

When dollar unit sampling is used, the sampling unit is defined as an individual dollar in the accounting population sampled. For each dollar randomly selected, the transaction to which it belongs is audited and the resulting error, if any, is prorated to the sampled dollar. For instance if a transaction is found to be 50 percent overstated, then each dollar of the transaction is assigned a 50-cent overstatement error tainting.

When individual dollar units are sampled in an accounting population and the maximum possible overstatement error per dollar is M (usually $M = 100$ cents), there are $M + 1$ possible outcomes for the amount of overstatement error in a sample dollar when the errors are measured to the nearest cent: $0, 1, 2, \ldots, M$. The population proportions of dollars with such errors are denoted by $p_0, p_1, p_2, \ldots, p_M$, where $p_i > 0$ $(i = 0, 1, \ldots, M)$ and $\sum p_i = 1$. If Y denotes the number of dollar units in the population, then the population total overstatement error D (in dollars) is given by:

$$D = (Y/100) \sum_{i=1}^{M} i p_i . \tag{1}$$

When a random sample of n dollar units is randomly selected from the population with replacement, the observed counts $w_0, w_1, w_2, \ldots, w_M$ for the $M + 1$ different error amounts follow the multinomial probability distribution, given $\vec{p} = (p_0, p_1, \ldots, p_M)$ and n, as illustrated in Neter, Leitch, and Fienberg (1978) for an accounting population. Fienberg et al. (1977) proposed a bound for the total overstatement error D obtained by first developing a $1 - \alpha$ multidimensional confidence region for the multinomial parameters $\vec{p}$ based on the observed counts $\vec{w} = (w_0, w_1, \ldots, w_M)$, i.e., a confidence region comprised of the $\vec{p}$ satisfying:

$$\sum_{S} \frac{n!}{z_0! z_1! \ldots z_M!} \prod_{i=0}^{M} p_i^{z_i} > \alpha, \tag{2}$$

where S is the set of those outcomes $\vec{z} = (z_0, z_1, \ldots, z_M)$ which are deemed to be "as extreme as or less extreme than" the observed result $\vec{w}$. The upper multinomial bound B_U is then obtained by maximizing the function (1) over the confidence region established by (2).

The set S can be defined in different ways. Fienberg et al. (1977) proposed the use of the "step-down" S-set for developing the confidence region in (2) because it has heuristic appeal and facilitates computations. Essentially, the step-down S-set consists of all outcomes for which the following two criteria are met:

(a) The total number of errors does not exceed the observed number of errors.

(b) Any individual error does not exceed the corresponding observed error.

For this step-down S-set, it was shown that the only parameters entering the maximization are: (1) those corresponding to the K observed errors $e_1, e_2, \ldots, e_K$—i.e., $p_{e_1}, p_{e_2}, \ldots, p_{e_K}$—and (2) p_0 and p_M. For convenience, e_1 shall denote the smallest error tainting in the sample and e_K the largest tainting. Consequently, the less extreme outcomes as defined for the step-down S-set are those which meet the condition:

$$\sum_{i=j}^{K} z_{e_i} < K - j + 1, \qquad j = 1, \ldots, K.$$

TABLE 1

Step-Down S-Set for Two Errors (Sample Size = n)

Step-down S-set			Total Nonzero Outcomes	Total Error
z_0	z_{10}	z_{20}	Criterion (a)	Criterion (b)
$n-2$	1	1	2	30
$n-2$	2	0	2	20
$n-1$	0	1	1	20
$n-1$	1	0	1	10
n	0	0	0	0

As an example, suppose a simple random sample of size n is selected and two errors are observed, 10 cents and 20 cents. Under criterion (a) no outcome in the step-down S-set can have more than two errors, and under criterion (b) the sum of the errors for each outcome in the S-set cannot exceed 30 cents, i.e. an outcome of two 20-cent errors is not allowed. Further, the cumulative frequency of errors cannot exceed 1 for the 20-cent outcome and 2 for the 10-cent outcome. As a result, the outcomes in the step-down S-set for this example can be represented by the first three columns of Table 1. As indicated by the last two columns of Table 1, the two criteria which establish the formation of the step-down S-set are both satisfied.

The formulation for B_U, incorporating the step-down S-set of Fienberg et al. (1977) is then:

$$B_U = \text{Max}(Y/100)(e_1 p_{e_1} + \cdots + e_K p_{e_K} + e_M p_M) \tag{3}$$

subject to

$$p_0 + \sum_{j=1}^{K} p_{e_j} + p_M = 1, \tag{3a}$$

$$\sum_{i=j}^{K} z_{e_i} \leqslant K - j + 1, \qquad j = 1, \ldots, K, \tag{3b}$$

$$\sum_{S} \frac{n!}{z_0! z_{e_1}! \cdots z_{e_K}!} \left(p_0^{z_0} p_{e_1}^{z_{e_1}} \cdots p_{e_K}^{z_{e_K}} \right) \geqslant \alpha, \tag{3c}$$

$$p_0, p_{e_j}, p_M \geqslant 0, \qquad j = 1, \ldots, K. \tag{3d}$$

The obtaining of the multinomial bound with the step-down S-set (3b) has turned out to be computationally feasible whereas some other possible definitions of "as extreme as or less extreme than" (e.g., the dollar amount S-set supported by Teitlebaum, McCray and Leslie 1978) involve very great computational difficulties. While the step-down S-set does not provide a complete ordering of the sample outcomes and consequently under some circumstances has a confidence level less than the nominal level, extensive simulation studies by Leitch et al. (1982) have shown that the confidence levels with the step-down S-set usually exceed the nominal level or are very close to it (e.g., 92.4 percent for a nominal 95 percent level; see Leitch et al. (1982) for details).

The Lower Multinomial Bound

The formulation for the lower multinomial bound B_L on total overstatement errors[4] differs in three important respects from that for the upper bound. First, the S-set used to establish the multidimensional confidence region for $\vec{p}$ consists of outcomes $\vec{z}$ which are deemed to be "as extreme as or *more* extreme than" the observed result $\vec{w}$. Second, B_L is obtained by *minimizing* D over the confidence region. Third, and in a manner similar to that followed by Fienberg et al. (1977), it can be shown that the only parameters entering the minimization are those corresponding to the K observed errors in the sample.

One way to define the more extreme outcomes for the lower bound S-set is to take the complement of the outcomes associated with the upper bound S-set. However, since the step-down S-set used for obtaining B_U does not provide a complete ordering of all sample outcomes, use of the complement of the step-down S-set would consider some outcomes as more extreme that are not.

Consequently, a "step-up" S-set is here proposed for developing the confidence region to obtain B_L. The step-up S-set consists of outcomes for which the following two criteria hold:

(A) The total number of errors is at least as great as the observed number of errors.

(B) Any individual "more extreme" error cannot be less than the corresponding observed error.

Consequently, the more extreme outcomes as defined for the step-up S-set are those which meet the condition:

$$\sum_{i=j}^{K} z_{e_i} \geq K - j + 1, \qquad j = 1, \ldots, K. \tag{4}$$

Table 2 illustrates the step-up S-set for the case where two errors, 10 and 20 cents, are observed in a sample of n.

TABLE 2

Step-Up S-Set for Two Errors (Sample Size = n)

Step-up S-set			Total Nonzero Outcomes	Total Error
z_0	z_{10}	z_{20}	Criterion (A)	Criterion (B)
$n-2$	1	1	2	30
$n-2$	0	2	2	40
$n-3$	0	3	3	60
$n-3$	1	2	3	50
$n-3$	2	1	3	40
$n-4$	0	4	4	80
⋮	⋮	⋮	⋮	⋮
0	$n-1$	1	n	$10(n-1)+20$

[4] The discussion in this paper will focus on overstatement errors. A lower bound on understatement errors is obtained in corresponding fashion. The multinomial bound approach does not lend itself to net errors, where overstatement and understatement errors are combined algebraically. See Neter et al. (1978) for procedures for treating combined overstatement and understatement errors.

The formulation of B_L, incorporating the step-up S-set, is then:

$$B_L = \text{Min}(Y/100)(e_1 p_{e_1} + \cdots + e_K p_{e_K}) \tag{5}$$

subject to

$$p_0 + \sum_{j=1}^{K} p_{e_j} = 1, \tag{5a}$$

$$\sum_{i=j}^{K} z_{e_i} \geq K - j + 1, \qquad j = 1, \ldots, K, \tag{5b}$$

$$\sum_{S} \frac{n!}{z_0! z_{e_1}! \cdots z_{e_K}!} (p_0^{z_0} p_{e_1}^{z_{e_1}} \cdots p_{e_K}^{z_{e_K}}) \geq \alpha, \tag{5c}$$

$$p_0, p_{e_j} \geq 0, \qquad j = 1, \ldots, K. \tag{5d}$$

Unlike the step-down S-set used to obtain B_U, the number of terms in the step-up S-set used to evaluate constraint (5c) is dependent upon the sample size (compare the S-sets in Tables 1 and 2). For sample sizes of 100 to 500, which are frequently used in auditing applications, the number of terms required to represent the step-up S-set is very large. As a result, the costs of computer processing time and storage required to evaluate the probability sum in (5c) is too great for a direct computation of the lower multinomial bound for all but very simple cases.

Nesting of Probability Terms

In this section we describe how nesting of probability terms is used to eliminate the effect of sample size, for a given number of sample errors, on the number of terms in the step-up S-set. This nesting approach is similar to that discussed in Plante (1980) and used by Leitch et al. (1982) for the upper bound; the objective there, however, was to reduce calculations for samples containing relatively high error rates.

The nesting of probability terms simplifies the computation of (5c) by considering the S-set to be made up of those outcomes that satisfy criterion (A), less the outcomes within (A) that fail to satisfy criterion (B). The implementation of nesting for the lower bound involves two steps: (1) relaxing the restrictions imposed by criterion (B), and (2) subtracting out those terms resulting from this relaxation that violate criterion (B).

Relaxation of Criterion (B)

The relaxation of the S-set restriction (5b) is done in such a way that the left-hand side of the confidence region inequality in (5c) becomes:

$$\sum_{i=k}^{n} \binom{n}{i} p_0^{n-i} P^i = 1 - \sum_{i=0}^{K-1} \binom{n}{i} p_0^{n-i} P^i, \qquad \text{where} \tag{6}$$

P = The sum of the multinomial parameters corresponding to the observed sample errors.

For the example in Table 2,

$K = 2$,

$P = (p_{10} + p_{20})$.

The influence of sample size has been removed in the right-hand side of (6) since this expression requires only K terms.

Subtraction of Excess Terms

Those terms in (6) that violate criterion (B) must now be subtracted. These terms may be obtained by considering the expansion of P^i in (6) for each i and enumerating all the possible violations of criterion (B) that result from this expansion. As an illustration, consider the expansion of P^2 for the example used in Table 2. This expansion would result in the following terms for (6),

$$\binom{n}{2} p_0^{n-2} P^2 = \binom{n}{2} p_0^{n-2} (p_{10} + p_{20})^2$$

$$= \binom{n}{2} p_0^{n-2} p_{10}^2 p_{20}^0 + 2\binom{n}{2} p_0^{n-2} p_{10} p_{20} + \binom{n}{2} p_0^{n-2} p_{10}^0 p_{20}^2 . \quad (7)$$

The first term in (7) violates (B) since it involves an outcome of zero for a 20-cent error.

In general, each of the violation terms can be partitioned into two distinct factors Q^m and V_l, as follows

$$\frac{n!}{(n-l-m)!m!f_l} p_0^{n-l-m} Q^m V_l , \quad \text{where} \quad (8)$$

l = A number which represents the cumulative frequency for parameters in P that violate (5b). For instance, if for some parameter j, the cumulative frequency of the last $(K-j+1)$ parameters violates (5b), then:

$$l = \sum_{i=j}^{K} z_{e_i} \quad \text{and} \quad (8a)$$

$$f_l = \prod_{i=j}^{K} z_{e_i}! ; \quad (8b)$$

m = A number which represents the cumulative frequency of the first $(j-1)$ parameters in P that do not violate (5b), such that

$$m = \sum_{i=1}^{j-1} z_{e_i} \quad \text{and} \quad K - l \leqslant m \leqslant n - l;$$

V_l = A product of the $(K-j+1)$ multinomial parameters whose cumulative frequency of occurrence (sum of exponents) is l;

Q = A sum of $(j-1)$ multinomial parameters whose cumulative frequency of occurrence is m.

To illustrate (8) for the violation term in example (7), $l = 0$, $f_l = 0!$, $V_l = p_{20}^0$, $m = 2$, and $Q = p_{10}$.

The sum of the violation terms corresponding to a particular V_l can be expressed as follows:

$$\sum_{m=K-l}^{n-l} \frac{n!}{(n-l-m)!f_l m!} p_0^{n-l-m} Q^m V_l = C_l V_l \sum_{m=K-l}^{n-l} \binom{n-l}{m} p_0^{n-l-m} Q^m \quad (9)$$

which equals

$$C_l V_l \left[(p_0 + Q)^{n-l} - \sum_{m=0}^{K-l-1} \binom{n-l}{m} p_0^{n-l-m} Q^m \right] \quad \text{where} \quad (9a)$$

$$C_l = \frac{n!}{(n-l)!f_l} . \quad (9b)$$

The number of terms in (9a) is $(K-l+1)$ and thus the influence of sample size on the number of terms has been effectively removed. Defining V as the set of all possible V_l's resulting from the expansion of (6), constraint (5c) can now be written using (6) and (9a) as follows,

$$1-\sum_{i=0}^{K-1}\binom{n}{i}p_0^{n-i}p^i-\sum_V C_l V_l\left[(p_0+Q)^{n-l}-\sum_{m=0}^{K-l-1}\binom{n-l}{m}p_0^{n-l-m}Q^m\right]>\alpha. \quad (10)$$

Further computational simplifications are possible since many of the $C_l V_l$ combinations share a common multiplier of the form

$$(p_0+Q)^{n-l}-\sum_{m=0}^{K-l-1}\binom{n-l}{m}p_0^{n-l-m}Q^m. \quad (11)$$

It can be shown that the number of distinct multipliers is exactly $(K-1)$, one less than the number of nonzero errors.

The number of terms required to represent (10) increases substantially as the numbers of errors increase. For instance, 6 errors require 29 terms but 14 errors require 290,526 terms. Indeed, for samples containing more than ten errors the number of terms in (10) becomes too great for a feasible computer implementation.

Requirements of the Optimization Algorithm

Prior to a study of the effectiveness of the lower multinomial bound, we consider first the requirements of the nonlinear optimization algorithm used to obtain the lower bound. The reduced gradient algorithm described in Plante (1980) for the upper multinomial bound is also used to obtain the lower multinomial bound. As is the case for the upper bound, (5c) is binding for the lower bound in the optimal solution. This can be shown in a manner similar to that followed by Leitch et al. (1980) for the upper bound. Thus, the lower multinomial bound model can be reformulated to adhere to the binding constraint requirement of the reduced gradient algorithm without the addition of slack or artificial variables.

The reduced gradient approach also requires an initial feasible solution. This is easily obtained for the upper bound, but an initial feasible solution for the lower bound model is not immediately obvious. Consequently a phase I-phase II procedure utilizing artificial variables has been added to the algorithm for the lower bound. An effective initial starting point for the lower bound utilizes the artificial variable p_A, where

$$\begin{aligned} p_A &= 1-p_0, \\ p_0 &= (1-\alpha)^{1/n}, \\ p_i &= 0, \qquad i=1,\ldots,K. \end{aligned} \quad (12)$$

Effectiveness of the Lower Multinomial Bound

Method of Analysis

Similar to the analyses performed for the upper multinomial bound by Fienberg et al. (1977) and Leitch et al. (1982), the effectiveness of the lower multinomial bound is assessed with respect to how much larger (less conservative) is the lower multinomial bound than the Stringer bound. The measure of effectiveness used for this comparison is the lower multinomial bound expressed as a percentage of the Stringer bound. Comparisons of the lower multinomial bound and the Stringer bound are made for the error distributions used in Leitch et al. (1982) for studying the upper multinomial bound. These distributions are commonly found in accounting populations, as re-

TABLE 3

Comparison of the Lower Multinomial Bound and the Stringer Bound
(Sample Size = 100, Population Size = 1,000,000, Confidence Level = 95%)

Distribution	Number of Errors	Multinomial Bound (1000's)	Stringer Bound (1000's)	Multinomial Bound as a Percent of Stringer Bound (%)
J	6	2.5	1.2	208.3
	8	4.1	1.9	215.8
	10	5.8	2.9	200.0
J-100	6	5.5	2.2	250.0
	8	7.8	3.3	236.4
	10	9.8	4.4	222.7
Unimodal	6	9.3	5.0	186.0
	8	14.5	8.2	176.8
	10	19.7	11.4	172.9

ported by Johnson, Leitch and Neter (1981):

(1) *Reversed J shaped*—Most of the errors are concentrated near zero (denoted as *J*).

(2) *Reversed J shaped with* 100 *cent errors*—Most of the errors are concentrated near zero plus some 100-cent errors (denoted as $J - 100$).

(3) *Unimodal shaped*—Most of the errors are concentrated away from zero about a positive mean.

Cases of 6, 8 and 10 sample errors that follow the pattern of each of these distributions were studied; see Leitch et al. (1982) for actual error patterns and findings from a simulation study that comparisons between average bounds obtained with repeated sampling from the same population are very similar to comparisons of bounds based on a single sample containing errors following the pattern of the population. Table 3 presents the lower multinomial and Stringer bounds for the various error patterns, each based on a population book amount of 1 million dollars, sample size of 100 and confidence level of 95 percent.

Findings

The results in Table 3 indicate that the lower multinomial bound is much less conservative than the Stringer bound for all cases studied. Indeed the smallest ratio of the lower multinomial bound to the Stringer bound is 172.9 percent and the largest is 250 percent. The error distributions exhibiting the largest relative gains by the lower multinomial bound are the *J* and $J - 100$ distributions while the smallest relative gains are achieved for the unimodal error distribution. The results in Table 3 also demonstrate that, for the cases studied, the relative gain by the lower multinomial bound over the Stringer bound decreases as the number of sample errors increases.[5]

Clustering of Sample Errors

Since the lower multinomial bound is much less conservative than the Stringer bound for the cases studied, it would be desirable to extend the applicability of the lower multinomial bound to samples containing more than ten errors.[6] This can be done by clustering errors, which reduces the number of terms required in (10) for a

[5] When absolute differences are considered, the gains with the lower multinomial bound are largest for the unimodal distribution and the absolute differences increase with the number of errors.

[6] Some simulation results suggest that the large-sample confidence limits for ratio and difference estimators with line-item sampling and for the mean-per-unit estimator with dollar unit sampling may at times not work well even with 20 or 25 errors in the sample.

given number of sample errors at the expense of a more conservative lower bound. The procedure is similar to the clustering procedure used by Leitch et al. (1982) for the upper multinomial bound.

For the lower bound each error within a cluster assumes the value of the *smallest* error in that cluster. For example, if a 10-cent error and a 20-cent error are clustered into one cluster the resulting cluster would be considered to contain two 10-cent errors. Assigning the value of the smallest error in a cluster to all errors in the cluster insures that the resulting lower bound approximation is no larger than that obtained for unclustered errors. This can be shown in a manner similar to that followed by Leitch et al. (1980) for the upper bound.

For clustered errors the formation of the step-up S-set is easily defined by expressing criteria (A) and (B) as

$$\sum_{i=j}^{m} z_{s_i} \geq \sum_{i=j}^{m} c_i, \qquad j = 1, \ldots, m, \qquad \text{where} \tag{13}$$

m = The number of clusters,
c_i = The number of errors in the ith cluster,
s_i = The value assumed by errors in the ith cluster, such that $s_m > s_{m-1} > \cdots > s_1$.

The clustering of sample errors substantially reduces the number of terms required by the step-up S-set. For instance, the grouping of 15 sample errors into six clusters for the unimodal error distribution results in 1,475 terms in the step-up S-set. In contrast, the step-up S-set for only 14 unclustered errors requires 290,526 terms.

The algorithm used to cluster sample errors for the lower multinomial bound is the same one used for the upper multinomial bound, namely that proposed by Fisher (1958). It is here used to determine, for a given number of clusters, m, a clustering which minimizes the criterion,

$$C = \sum_{j=1}^{m} \sum_{i=1}^{K} (x_{ij} - s_j) \qquad \text{where} \tag{14}$$

s_j = The smallest error in the jth cluster,
x_{ij} = The ith error in the jth cluster, such that

$$s_j \leq x_{ij} < s_{j+1}. \tag{14a}$$

Effectiveness of the Lower Multinomial Bound for Clustered Errors

Method of Analysis

We again compare the lower multinomial bound for clustered errors with the Stringer bound using as the measure of effectiveness the lower multinomial bound as a percentage of the Stringer bound. The same error distributions as before are employed to study the effectiveness of the clustered bound. In addition, the uniform distribution is also employed since it is an extreme distribution as far as clustering of errors is concerned.[7] Up to 25 errors are studied and up to ten clusters are employed.

Tables 4 and 5 present the results of the analysis for 6 and 10 errors, and Table 6 extends the results to 15, 20 and 25 errors. These tables also show the amount of computer processing time on a CDC-70/74 required to obtain the clustered lower multinomial bound.

[7]The uniform distribution is not commonly encountered in accounting problems according to the Johnson et al. (1981) study.

TABLE 4

Effect of Clustering on the Lower Multinomial Bound for Total Overstatement Error (Sample Size = 100, Population Size = 1,000,000, Confidence Level = 95%, Number of Errors = 6)

Uniform Distribution						
# of Clusters	1	2	3	4	5	6
Mult. Bound (1000's)	2.1	8.2	11.7	13.5	14.5	15.9
% Unclustered Error Bound	13.2	51.6	73.6	84.9	93.1	100.0
% Stringer Bound	21.4	83.7	119.4	137.8	151.0	162.2
CPU Time (Seconds)	0.0	0.8	1.1	1.0	1.5	1.8
Unimodal Distribution						
# of Clusters	1	2	3	4	5	6
Mult. Bound (1000's)	1.4	4.0	6.1	7.7	8.4	9.3
% Unclustered Error Bound	15.1	43.0	65.6	82.8	90.3	100.0
% Stringer Bound	28.0	80.0	122.0	154.0	168.0	186.0
CPU Time (Seconds)	0.1	0.9	2.1	1.7	1.6	1.6
***J* − 100 Distribution**						
# of Clusters	1	2	3	4	5	6
Mult. Bound (1000's)	0.6	1.3	3.6	4.7	5.5	na
% Unclustered Error Bound	10.9	23.6	65.5	85.5	100.0	na
% Stringer Bound	27.3	59.1	163.6	213.6	250.0	na
CPU Time (Seconds)	0.1	0.2	3.9	5.2	5.7	na
***J* Distribution**						
# of Clusters	1	2	3	4	5	6
Mult. Bound (1000's)	0.6	0.8	1.8	2.2	2.5	na
% Unclustered Error Bound	24.0	32.0	72.0	88.0	100.0	na
% Stringer Bound	50.0	66.7	150.0	183.3	208.3	na
CPU Time (Seconds)	0.1	0.2	4.4	3.9	3.5	na

TABLE 5

Effect of Clustering on the Lower Multinomial Bound for Total Overstatement Error (Sample Size = 100, Population Size = 1,000,000, Confidence Level = 95%, Number of Errors = 10)

Uniform Distribution

# of Clusters	1	2	3	4	5	6	7	8	9	10
Mult. Bound (1000's)	2.8	14.9	21.6	25.3	27.8	29.2	30.2	31.0	31.7	33.0
% Unclustered Error Bound	8.4	45.2	65.5	76.7	84.2	88.5	91.5	93.9	96.1	100.0
% Stringer Bound	12.8	68.3	99.1	116.1	127.5	133.9	138.5	142.2	145.4	151.3
CPU Time (Seconds)	0.0	2.1	2.4	2.5	4.3	5.8	7.2	13.7	28.8	99.3

Unimodal Distribution

# of Clusters	1	2	3	4	5	6	7	8	9	10
Mult. Bound (1000's)	2.2	7.8	11.7	14.0	15.8	17.4	18.1	18.6	19.2	19.7
% Unclustered Error Bound	11.2	39.6	59.4	71.1	80.2	88.3	91.9	94.4	97.5	100.0
% Stringer Bound	19.3	68.4	102.6	122.8	138.6	152.6	158.8	163.2	168.5	172.9
CPU Time (Seconds)	0.0	2.1	2.9	5.0	7.3	10.7	22.4	28.7	82.2	107.4

***J* − 100 Distribution**

# of Clusters	1	2	3	4	5	6	7	8	9	10
Mult. Bound (1000's)	0.8	1.8	5.1	6.9	8.4	9.2	9.7	9.8	na	na
% Unclustered Error Bound	8.2	18.4	52.0	70.4	85.7	93.9	99.0	100.0	na	na
% Stringer Bound	18.1	40.9	115.6	156.8	190.9	209.1	220.5	222.7	na	na
CPU Time (Seconds)	0.0	0.3	13.0	12.7	18.5	23.2	41.3	38.8	na	na

***J* Distribution**

# of Clusters	1	2	3	4	5	6	7	8	9	10
Mult. Bound (1000's)	0.8	1.8	3.3	4.1	4.9	5.3	5.6	5.8	na	na
% Unclustered Error Bound	13.7	31.0	56.9	70.7	84.5	91.4	96.6	100.0	na	na
% Stringer Bound	27.6	62.1	113.8	141.4	169.0	182.8	193.1	200.0	na	na
CPU Time (Seconds)	0.0	0.2	9.3	9.9	8.2	14.9	10.9	14.2	na	na

TABLE 6

*Comparison of the Lower Multinomial Bound for Clustered Errors and the Stringer Bound (Sample Size = 100, Population Size = 1,000,000, Confidence Level = 95%, Number of Clusters = 6)**

Distribution	Number of Errors	Modified Multinomial Bound (1000's)	Stringer Bound (1000's)	Multinomial Bound as a Percent of Stringer Bound (%)	CPU Time (seconds)
J	15	8.4	5.3	158.5	36.4
	20	12.0	8.3	144.6	155.4
	25	14.4	11.6	124.1	159.8
J – 100	15	17.6	10.8	163.0	66.0
	20	21.3	14.2	150.0	134.0
	25	28.7	22.5	127.6	229.8
Unimodal	15	27.9	20.9	133.5	122.5
	20	37.7	30.9	122.0	466.7
	25	44.7	41.6	107.5	177.6
Uniform	15	47.1	38.9	121.1	86.1
	20	65.8	57.3	114.8	172.4
	25	79.6	76.5	104.1	129.9

*Five clusters are used to obtain modified multinomial bounds for the 25 error cases.

Findings

The results presented in Tables 4 and 5 demonstrate the effectiveness of the clustered lower multinomial bound when there are ten or fewer errors in the sample. Even when only four clusters are used, the lower multinomial bound is larger (less conservative) than the Stringer bound in all cases. Use of more clusters improves the comparative performance of the lower multinomial bound still further. However, for a given number of clusters, as the number of sample errors increases the relative gain by the clustered lower multinomial bound over the Stringer bound decreases rapidly. For instance, the use of five clusters for 6 and 10 errors that are uniformly distributed yields lower multinomial bounds that are 151% and 127.5% of the Stringer bounds, respectively.

The results for 15 and 20 errors and six clusters and for 25 errors and five clusters in Table 6 indicate that the clustered multinomial bound continues to be comparatively effective for larger numbers of errors. Indeed for the 20-error cases, the lower multinomial bound is at least 114.8% and as much as 150% of the Stringer bound and for the 25-error cases the ratio varies from 104.1% to 127.6%.

At the present time, computer processing time is too great for an economical application of the lower multinomial bound for more than 25 sample errors grouped into a sufficiently large number of clusters so that substantial gains over the Stringer bound can be achieved. The CPU time required for the lower multinomial bound in all cases studied exceeds that for calculating the upper bound, often by a factor of two or three. One reason is that the number of search iterations required tends to be larger for obtaining the lower bound than for obtaining the upper bound. Consequently the implementation of the lower multinomial bound model is not as practical for as large a number of sample errors as is the implementation of the upper bound model.

Summary

In this paper a methodology has been developed that allows the determination of lower multinomial bounds for the total error amount in an accounting population when there are up to 25 sample errors. The development of the methodology for the lower multinomial bound has been in terms of the determination of a lower confidence bound on the total overstatement error, but the determination of a lower confidence bound on the total understatement error is also possible. The information derived from a lower bound is necessary for determining the amount of adjustment required to be made in an account, tax return, or contract settlement in many public and private auditing situations.[8]

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